Differential Geometry IV: General Relativity

14.1 Let (Σ^3, g) be a smooth Riemannian manifold. We will say that (Σ, g) is *asymptotically flat* with n asymptotically flat ends if there exists a compact subset $\mathcal{K} \subset \Sigma$ such that $\Sigma \setminus \mathcal{K}$ has n connected components $\Sigma_1, \ldots, \Sigma_n$ and, for each of them, there exists a diffeomorphism $\Phi_i : \Sigma_i \to \mathbb{R}^3 \setminus B_1(0)$ with the following property: In the Cartesian coordinates (x^1, x^2, x^3) associated to this diffeomorphism, the components of the metric g satisfy for any $m \in \mathbb{N}$:

$$\partial^m (g_{ij} - \delta_{ij}) = O(r^{-m-1}) \quad \text{as} \quad r \to +\infty,$$

where $r = \left(\sum_{i=1}^3 (x^i)^2\right)^{\frac{1}{2}}$. For any asymptotically flat end Σ_l, we will define the ADM mass $(M_{ADM})_l$ as the limit (in these coordinates)

$$M_{ADM} = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{S_r} \Big(\sum_{i,j=1}^{3} \big(\partial_j g_{ij} - \partial_i g_{jj}\big) N^i \, dA \Big),$$

where S_r is the coordinate sphere of radius r, N is the normal to S_r (with respect to the flat metric) and dA is the volume form on S_r induced by the flat metric.

(a) Show that the value of the ADM mass in each asymptotically flat end is invariant under coordinate transformations of the form $x \to x + c + F(x)$, where $c \in \mathbb{R}^3$ is a constant and $F : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$\partial^m F = O(r^{-m-1}) \quad \text{for all } m \in \mathbb{N}$$

(coordinates in this class are usually called *asymptotically Euclidean*).

(b) Show that the slice $\{t = 0\}$ in the maximally extended Schwarzschild spacetime with mass parameter $M > 0$, equipped with its induced metric, is asymptotically flat with two asymptotically flat ends. Show that the ADM mass of each end is equal to M.

*(c) Let $(\mathbb{R}^3; \bar{g}^{(\epsilon)}, k^{(\epsilon)})$ be a smooth family of initial data sets for the Einstein equations

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi\epsilon T_{\mu\nu}$$

with $\epsilon \geqslant 0$, such that $\bar{g}^{(0)}_{ij} = \delta_{ij}$ and $k^{(0)} = 0$ for all $\epsilon \geqslant 0$. Assume that $(\mathbb{R}^3; \bar{g}^{(\epsilon)})$ is asymptotically flat for all $\epsilon \geqslant 0$. Defining $h = \frac{d}{d\epsilon} \bar{g}^{(\epsilon)}\Big|_{\epsilon=0}$ to be the linearization of $\bar{g}^{\epsilon}$ around $\epsilon = 0$, show that

$$\sum_{i,j=1}^{3} \Big(-\partial_i^2 h_{jj} + \partial_i \partial_j h_{ij} \Big) = 16\pi T(\hat{n}, \hat{n})$$

(Hint: Compute the linearization of the Hamiltonian constraint equation.) Deduce that, if the energy momentum tensor T satisfies the positive energy condition $T(\hat{n}, \hat{n}) \geqslant 0$, then

$$\frac{d}{d\epsilon} M^{(\epsilon)}_{ADM}\Big|_{\epsilon=0} \geqslant 0.$$

Remark. The above is a special case of the following fundamental result, proven in two different ways by Schoen–Yau (1979) and Witten (1981):

The positive mass theorem: Let $(\Sigma^3, \bar{g}, k)$ be an asymptotically flat initial data set for the Einstein equations for a matter field satisfying the *dominant energy condition* (e.g. vacuum, scalar field, etc). Then the ADM mass of each asymptotically flat end satisfies $M_{ADM} \geqslant 0$, with equality if and only if $(\Sigma, \bar{g}, k)$ is a trivial initial data set, i.e. $\Sigma = \mathbb{R}^3$ and $(\bar{g}, k)$ are the induced metric and second fundamental form of a Cauchy hypersurface of Minkowski spacetime (if $k = 0$, this implies that $\bar{g}$ is the flat Euclidean metric).

***14.2** In this exercise, we will show that the $\dot{H}^1$ norm of solutions to the wave equation on Schwarzschild exterior remains uniformly bounded in time. For the rest of the exercise, let $(\mathcal{M}, g_M)$ be the maximally extended Schwarzschild spacetime of mass $M > 0$. Let also Σ_0 be a spacelike hypersurface such that we have $\Sigma_0 = \{t = 0\}$ in the region $r \geqslant R_0$ for some given $R_0 > 2M$ (with respect to the usual (t, r, θ, φ) coordinates on region I) and Σ_0 intersects $\mathcal{H}^+$ transversally. For convenience, we can choose Σ_0 to be rotationally invariant. We will also denote with V the stationary Killing vector field of Schwarzschild (with $V = \frac{\partial}{\partial t}$ in the usual (t, r, θ, φ) coordinates). We will also denote with $\Phi_t^{(V)}$ the (isometric) flow map of V (which is simply the time translation map in the usual coordinate system) and

$$\Sigma_\tau = \Phi_\tau(\Sigma_0).$$

(a) Draw how Σ_0 and Σ_τ, $\tau \geqslant 0$, look like on the Penrose diagram.

(b) Let Y be a future directed null vector field defined along $\mathcal{H}^+$ such that Y is invariant under the flow of V (hence, in any coordinate system in which V is a coordinate vector field, the components of Y are constant in the corresponding coordinates) and satisfies the normalization condition $\langle Y, V \rangle = -2$. Show that $\langle \nabla_V Y, V \rangle = 2\kappa > 0$, where κ depends only on M (this is known as the *surface gravity* of the event horizon). Show also that $\nabla_V V = \kappa V$ along $\mathcal{H}^+$ *Hint: Work in one of the coordinate systems that we have already seen are regular across $r = 2M$, for instance the $(t^*, r, \theta, varphi)$ coordinate system from Ex. 8.2.)*

(c) Let $\mathcal{D}$ denote the future domain of dependence of Σ_0 intersected with the region I. Extend the vector field Y constructed above on $\mathcal{D}$ in such a way that Y remains invariant under the flow of V, $Y = 0$ for $r \geqslant R_0$ and satisfies along $\mathcal{H}^+$: $\nabla_Y Y = -\sigma(V + Y)$ for some $\sigma > 0$ to be fixed in a moment. Set

$$N = V + Y.$$

Show that N is everywhere timelike on $\text{clos}\mathcal{D}$. In particular, deduce that, for any smooth function ψ, the following lower bound holds:

$$\int_{\Sigma_\tau} J_\mu^{(N)}[\psi] \hat{n}^\mu \geqslant C \int_{\Sigma_\tau} \left(|\bar{\nabla}\psi|^2 + |\hat{n}(\psi)|^2 \right) \text{dvol}_{\Sigma_\tau},$$

where $\hat{n}$ is the timelike future directed unit normal to $C > 0$ is independent of $\tau \geqslant 0$ and ψ. On the other hand, show that for the Killing vector field V we similarly have

$$\int_{\Sigma_\tau} J^{(V)}_\mu[\psi]\hat{n}^\mu \geqslant c_{r_0} \int_{\Sigma_\tau \cap \{r \geqslant r_0\}} \big(|\bar{\nabla}\psi|^2 + |\hat{n}(\psi)|^2\big)\, d\mathrm{vol}_{\Sigma_\tau}$$

for any $r_0 > 2M$ (but with c_{r_0} degenerating as $r_0 \to 2M$, in view of the fact that V becomes timelike there; we will say that the energy density associated to V *degenerates* at $\mathcal{H}^+$, while that associated to N is non-degenerate). Moreover, provided σ has been chosen large enough in terms of κ, the following bound holds on $\mathcal{H}^+$ for any smooth function ψ:

$$T_{\mu\nu}[\psi]\nabla^{(\mu}N^{\nu)} \geqslant c J^{(N)}_\mu[\psi]N^\mu,$$

where $c > 0$ is independent of ψ.

(d) Let us denote

$$E^{(V)}[\psi](\tau) \doteq \int_{\Sigma_\tau} J^{(V)}_\mu[\psi]\hat{n}^\mu, \quad E^{(N)}[\psi](\tau) \doteq \int_{\Sigma_\tau} J^{(N)}_\mu[\psi]\hat{n}^\mu.$$

Using the divergence identity for $J^{(V)}$, show that, for any smooth solution ψ of $\Box_g\psi = 0$ arising from compactly supported initial data on Σ_0, we have for any $r_0 > 2M$:

$$\sup_{\tau\geqslant 0} \int_{\Sigma_\tau \cap \{r\geqslant r_0\}} \big(|\bar{\nabla}\psi|^2 + |\hat{n}(\psi)|^2\big)\, d\mathrm{vol}_{\Sigma_\tau} \leqslant C_{r_0} E^{(V)}[\psi](0).$$

(e) Similarly, using the divergence identity for $J^{(N)}$ and the estimates established above for N, show that for any smooth solution ψ of $\Box_g\psi = 0$ arising from compactly supported initial data on Σ_0 and any $\tau \geqslant 0$:

$$E^{(N)}[\psi](\tau) + c\int_0^\tau \int_{\Sigma_\tau \cap \{r \leqslant R_0\}} \big(|\bar{\nabla}\psi|^2 + |\hat{n}(\psi)|^2\big)\, d\mathrm{vol}_{\Sigma_s}\, ds \leqslant C E^{(N)}[\psi](0) + C\int_0^\tau E^{(V)}[\psi](s)\, ds$$

for some constants $c, C > 0$ independent of τ, ψ (note that the spacetime integral above on the left hand side is trivially controlled by the left hand side in a region away from $r = 2M$; in the vicinity of $r = 2M$, one has to employ the properties established for $T_{\mu\nu}[\psi]\nabla^{(\mu}N^{\nu)}\big|_{r=2M}$ earlier). Deduce that, from the inequalities established earlier (for possibly different constants C, c):

$$E^{(N)}[\psi](\tau) + c\int_0^\tau E^{(N)}[\psi](s)\, ds \leqslant C\Big(E^{(N)}[\psi](0) + \tau E^{(V)}[\psi](0)\Big).$$

(f) Deduce that

$$\sup_{\tau\geqslant 0} E^{(N)}[\psi](\tau) < +\infty.$$